Lecture 4.

Completions.

If $(X, \mathcal{M}, \mu)$ is a measure space, and $E \in \mathcal{M}$ + $\mu(E) = 0$, then E is a <u>null set</u>. Thus, if $F \subseteq E$ <u>and $F \in \mathcal{M}$</u>, then $\mu(F) = 0$ so F is null. However, in general not all subsets of null sets need to be in $\mathcal{M}$.

<u>Def. 1</u>. A measure space $(X, \mathcal{M}, \mu)$ is complete if all subsets of null sets are measurable.

Every measure space can be completed.

<u>Thm</u>. Let $(X, \mathcal{M}, \mu)$ be measure space, $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and define $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M}, F \subseteq N \in \mathcal{N}\}$. Then, $\overline{\mathcal{M}}$ is a σ-algebra and there exist unique, complete measure $\bar{\mu}$ on $\overline{\mathcal{M}}$ s.t. $\bar{\mu} = \mu$ on $\mathcal{M}$.

<u>Pf.</u> Note that $\mathcal{N}$ is closed under countable unions. $\Rightarrow$ $\overline{\mathcal{M}}$ is closed under countable unions. To see that $\overline{\mathcal{M}}$ is also closed under complements, consider $E \cup F$ as above. Let $F \subseteq N \in \mathcal{N}$. WLOG assume $E \cap N = \emptyset$. (if not, $N \to N \setminus E$, $F \to F \setminus E$)

$$\Rightarrow (E \cup F)^c = \underbrace{(E^c \cap N^c)}_{\in \mathcal{M}} \cup \underbrace{(N \setminus F)}_{\subseteq N}$$

$\Rightarrow \overline{\mathfrak{M}}$ closed under complements

$\Rightarrow \overline{\mathfrak{M}}$ σ-algebra.

To define $\bar{\mu}$, we set

$\bar{\mu}(E \cup F) = \mu(E)$.

Well defined? Suppose

$E \cup F = E' \cup F'$, $F \subseteq N \in \mathfrak{N}$, $F' \subseteq N' \in \mathfrak{N}$.

$\Rightarrow E \subseteq E' \cup N' \Rightarrow \mu(E) \leq \mu(E')$

and similarly $\mu(E') \leq \mu(E) \Rightarrow$

$\mu(E') = \mu(E)$.

Checking $\bar{\mu}$ is complete measure is HW for next week. $\boxtimes$

Outer measures.

This is a useful tool for constructing measures.

Def. 2 An *outer measure* is a function $\mu^*: P(X) \to [0,\infty)$ s.t.

(i) $\mu^*(\emptyset) = 0$

(ii) $E \subseteq F \Rightarrow \mu^*(E) \leq \mu^*(F)$

(iii) $\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$.

Thm 2. Let $\mathcal{E} \subseteq P(X)$ be s.t. $\emptyset, X \in \mathcal{E}$ and $\rho: \mathcal{E} \to [0,\infty]$ s.t. $\rho(\emptyset) = 0$. Set

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \rho(E_k) : A \subseteq \bigcup_{k=1}^{\infty} E_k, \ E_k \in \mathcal{E}\right\}$$

Then, $\mu^*: P(X) \to [0,\infty]$ is an outer measure.

Pf: Clearly, $\mu^*(\emptyset) = 0$ since $\emptyset \subseteq \emptyset$.
Monotonicity is similarly trivial, if $A \subseteq B$, $B \subseteq \bigcup_1^\infty E_n \Rightarrow A \subseteq \bigcup_1^\infty E_n \Rightarrow$
$\mu^*(A) \le \sum_1^\infty \rho(E_n) \Rightarrow \mu^*(A) \le \mu^*(B)$
$= \inf\{ \quad \}$.

To check subadditivity, let $\{E_n\}_{n=1}^\infty$ be subcollection of $\mathcal{E}$ s.t. $A \subseteq \bigcup_1^\infty E_n$.
Let $\varepsilon > 0$ and choose $\{E_{n,l}\}_{l=1}^\infty$ be in $\mathcal{E}$ s.t. $\mu^*(E_n) \ge \sum_{l=1}^\infty \rho(E_{n,l}) - 2^{-n+1}\varepsilon$

By def, since $A \subseteq \bigcup_{n,l} E_{n,l}$
$$\mu^*(A) \le \sum_{n,l} \rho(E_{n,l}) \le \sum_{n=1}^\infty \left(\mu^*(E_n) + 2^{-n+1}\varepsilon\right)$$
$$= \sum_{n=1}^\infty \mu^*(E_n) + \varepsilon.$$
Since $\varepsilon > 0$ arb. $\Rightarrow$ subadditivity. $\blacksquare$